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Painlevé type equations and Hitchin systems[1]

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Abstract

In this survey we present the interpretation of isomondromy preserving equations on Riemann surfaces with marked points as reduced Hamiltonian systems. The upstairs space is the space of smooth connections of GL(N) bundles with simple poles in the marked points. We discuss relations of these equations with the Whitham quantization of the Hitchin systems and with the classical limit of the Knizhnik-Zamolodchikov-Bernard equations. The main example is the one-parameter family of Painlevé VI equation and its multicomponent generalization.

1 Introduction

The famous Painlevé VI equation depends on four free parameters ($PVI_{\alpha,\beta,\gamma,\delta}$) and has the form

$$\frac{d^2X}{dt^2} = \frac{1}{2}\left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-t}\right)\left(\frac{dX}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{X-t}\right)\frac{dX}{dt} +$$
$$+ \frac{X(X-1)(X-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{X^2} + \gamma\frac{t-1}{(X-1)^2} + \delta\frac{t(t-1)}{(X-t)^2}\right). \tag{1.1}$$

It was discovered by B.Gambier [1] in 1906. He accomplished the Painlevé classification program of the second order differential equations whose solutions have not movable critical points. This equation and its degenerations $PV - PI$ have a lot of applications in classical and quantum integrable systems (see, for example [2]), topological field theories [3], general relativity [4, 5], and in the Seiberg-Witten theory [6]. In this paper we discuss two important and interrelated aspects of PVI:

- PVI and isomonodromic deformations of linear differential equations;
- The Hamiltonian structure of PVI.

The derivation of PVI equation as the preserving monodromy condition was given by R.Fuchs [7], while the Hamiltonian structure of PVI was introduced by J.Malmquist [8].

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We incorporate the one-parameter family $\mathrm{PVI}_{\frac{\nu^2}{4},-\frac{\nu^2}{4},\frac{\nu^2}{4},\frac{1}{2}-\frac{\nu^2}{4}}=\mathrm{PVI}_\nu$ in a wide class of non-linear equations. They preserve monodromies of systems of linear equations on Riemann curves with marked points when the complex structures of curves are changed. These systems come from the flatness condition of vector bundles on the curves. We restrict ourself by considerations of smooth connections with simple poles only, and therefore don't include the Stokes phenomena. In such general form the isomondromy preserving equations were considered in [9]. Our investigation of these systems is inspired by methods developed in classical and quantum integrable systems. In general all the systems can be derived using three different constructions:
A)The symplectic reduction procedure from free infinite dimensional theory. This approach is very similar to the derivation of the Hitchin integrable systems [10];
B)The Whitham quantization of the Hitchin systems [11];
C)Classical limit of the Knizhnik-Zamolodchikov-Bernard (KZB) equations [12, 13].
We discuss these constructions separately and then demonstrate their application on the multi-component generalization of PVI_ν. The presentation is based for the most part on a previously published paper [14]. First, in Sect.2 we consider the elliptic form of PVI. In this form the relations of PVI with the Hitchin systems and KZB equations become transparent. Then we discuss these three approaches to the isomonodromic deformations. In Sect.6 the multicomponent generalization of PVI is described. Finally, we discussed some open problems related to PVI and its generalizations.

2 Elliptic form of PVI

1. Elliptization procedure
Soon after discovering of PVI (1.1) by Gambier, Painlevè presented it in terms of the Weierstrass elliptic functions [15]. This paper was almost forgotten for ninety years and the elliptic form was rediscovered recently in [16, 17]. We follow the derivation presented in [16], where the Hamiltonian form and symmetrices of PVI in terms of elliptic functions are treated.

Consider the family of elliptic curves

$$E_\tau = \mathbf{C}/(\mathbf{Z}+\mathbf{Z}\tau) \tag{2.1}$$

where $\tau \in H = \{Im\tau > 0\}$ Let $\wp(u|\tau)$ be the Weierstrass function

$$\wp(u| = \frac{1}{u^2} + \sum{}' \left(\frac{1}{(u+m\omega_1+n\omega_2)^2} - \frac{1}{(m\omega_1+n\omega_2)^2}\right),\ (\tau = \frac{\omega_2}{\omega_1}). \tag{2.2}$$

In the most part of the paper we put $\omega_1 = 1$ and $\wp(u|\tau) = \wp(u|1,\omega_2)$. $\wp(u|\tau)$ uniformize the elliptic curve

$$\wp_u(u|\tau) = 4(\wp(u|\tau)-e_1(\tau))(\wp(u|\tau)-e_2(\tau))(\wp(u|\tau)-e_3(\tau)), \tag{2.3}$$

$$e_i = \wp\left(\frac{T_i}{2}|\tau\right),\quad (T_0,\dots,T_3) = (0,1,\tau,1+\tau).$$

We consider two kind of transformations. First one is the lattice action

$$u \to u+m+n\tau,\ \tau\to\tau. \tag{2.4}$$

It leaves $\wp(u|\tau)$ and $\wp_u(u|\tau)$ invariant. The second is the modular transformation by $\mathrm{PSL}_2(\mathbf{Z})$

$$\wp\left(\frac{u}{c\tau+d}|\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2\wp(u|\tau),\quad \wp_u\left(\frac{u}{c\tau+d}|\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^3\wp_u(u|\tau). \tag{2.5}$$

Now consider another family of elliptic curves $E_t \to B$, $Y^2 = X(X-1)(X-t)$ parameterized by $B = \{t \in \mathbf{P}^1 \setminus (0, 1, \infty)\}$. There exists the morphism $\{E_\tau\} \to \{E_t\}$ defined as

$$(u, \tau) \to \left(X = \frac{\wp(u|\tau) - e_1}{e_2 - e_1}, Y = \frac{\wp_u(u|\tau)}{e_2 - e_1}, t = \frac{e_3 - e_1}{e_2 - e_1} \right). \tag{2.6}$$

Theorem 2.1 *In terms of (u, τ) $PVI_{\alpha,\beta,\gamma,\delta}$ takes the form*

$$\frac{d^2 u}{d\tau^2} = \partial_u U(u|\tau), \quad U(u|\tau) = \frac{1}{(2\pi i)^2} \sum_{j=0}^{3} \alpha_j \wp(u + \frac{T_j}{2}|\tau), \tag{2.7}$$

$$(\alpha_0, \ldots, \alpha_3) = (\alpha, -\beta, \gamma, \frac{1}{2} - \delta).$$

The proof of the equivalence of (1.1) and (2.7) is based on the Picard-Fuchs equation on elliptic curves. The Picard-Fuchs operator

$$L_t = t(t-1)\frac{\partial^2}{\partial t^2} + (1 - 2t)\frac{\partial}{\partial t} - \frac{1}{4}$$

acting on the holomorphic differential $\omega = (d_{E/B}x)/y$ yields the exact differential $\frac{1}{2} d_{E/B} \frac{y}{(x-t)^2}$. The Picard-Fuchs equation just means that periods of $d_{E/B}x/y$ are annihilated by L_t. Using the Picard-Fuchs operator Fuchs proved that PVI (1.1) is equivalent to the following equation

$$t(1-t)L_t \int_\infty^X d_{E/B}x/y = \left(\alpha + \beta \frac{t}{X^2} + \gamma \frac{t-1}{(X-1)^2} + (\delta - \frac{1}{2}) \frac{t(t-1)}{(X-t)^2} \right) Y, \tag{2.8}$$

The equivalence of (1.1) and (2.8) follows from the following equality

$$t(1-t)L_t \int_\infty^X d_{E/B}x/y = \frac{1}{2} \frac{t(t-1)Y}{(X-t)^2} + \frac{d^2 X}{dt^2} - \frac{1}{2} \left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-t} \right) \left(\frac{dX}{dt} \right)^2 +$$

$$\left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{X-t} \right) \frac{dX}{dt}, \quad Y^2 = X(X-1)(X-t).$$

The proof is straightforward.

Thus, PVI can be written in the form of the so-called μ-equation [16]

$$L_t \int_\infty^X \omega = s_{(\alpha,\beta,\gamma,\delta)}(X), \tag{2.9}$$

where the right hand side is a special section of the bundle E_t. It can be fixed by the symmetries of the equation.

Under the morphism (2.6) the holomorphic differential $d_{E/H}z$ on E_τ is transformed in $d_{E/B}x/y$, and $\frac{d^2}{d\tau^2}$ in L_t. More exactly, the left hand side of (2.9) takes the form

$$\frac{2\pi^2}{(e_1 - e_2)(e_1 - e_3)(e_2 - e_1)^{\frac{1}{2}}} \frac{d^2}{d\tau^2} \int_0^{u(\tau)} dz.$$

Taking into account that

$$Y = \frac{1}{2}(e_2 - e_1)^{-3/2} \wp_u(u, \tau)$$

we come finally to (2.7).

2. Hamiltonian structure

The hamiltonian form of (2.7) is defined by the standard symplectic form

$$\omega^{(0)} = \delta v \delta u, \tag{2.10}$$

and the Hamiltonian

$$H = \frac{v^2}{2} - U(u|\tau). \tag{2.11}$$

Consider the bundle $\mathcal{P}$ over the moduli space $\mathcal{M} = H/\mathrm{PSL}_2(\mathbf{Z})$ with the symplectic fibers parameterized by the local coordinates (v, u). It plays role of the extended phase space for the non-autonomous hamiltonian system (2.10),(2.11). The equation of motion (2.7) can be derived from the action $\mathcal{F}$ on $\mathcal{P}$

$$\delta\mathcal{F} = v\delta u - H\delta\tau. \tag{2.12}$$

The symmetries of the non-autonomous hamiltonian systems are determined by the invariance of the two-form ω on $\mathcal{P}$

$$\omega = \omega^{(0)} - \delta H \delta\tau = \delta v \delta u - \delta H \delta\tau. \tag{2.13}$$

It follows from (2.2) and (2.5) that the symmetry group is the semi-direct product of $\mathbf{Z}+\mathbf{Z}\tau$ and the group $\Gamma(2) \subset \mathrm{PSL}_2(\mathbf{Z})$. We consider a simplified version of this action in Sect.7 in detail.

3. Calogero-Inozemtsev equation and PVI

Let us introduce the new parameter κ and instead of (2.13) consider

$$\omega = \omega_0 - \frac{1}{\kappa}\delta H \delta\tau. \tag{2.14}$$

It can be achieved by the rescaling the dynamical variables (v, u) and periods ω_1, ω_2 $v \to \kappa^{-\frac{1}{2}}$, $u \to \kappa^{\frac{1}{2}}$, $\omega_1 \to \kappa^{\frac{1}{2}}$, $\omega_1 \to \kappa^{\frac{1}{2}}$. Then, (2.7) takes the form

$$\kappa^2 \frac{d^2 u}{d\tau^2} = -\partial_u U(u|\tau). \tag{2.15}$$

Put $\tau = \tau_0 + \kappa t^H$ and consider the system in the limit $\kappa \to 0$. We come to the equation

$$\frac{d^2 u}{(dt^H)^2} = -\partial_u U(u|\tau_0) \tag{2.16}$$

corresponding to the autonomous Hamiltonian system with the time-independent potential $U(u|\tau_0)$. It is just the rank one elliptic Calogero-Inozemtsev equation ($CI_{\alpha,\beta,\gamma,\delta}$) [18, 19]. The potential $U(u|\tau_0)$ was considered first by Darboux [20]. It arises also in the soliton theory [21]. Thus, we have in this limit

$$PVI_{\alpha,\beta,\gamma,\delta} \stackrel{\kappa\to 0}{\longrightarrow} CI_{\alpha,\beta,\gamma,\delta}. \tag{2.17}$$

There is the inverse procedure (the Whitham quantization) that allows to construct approximations of non-autonomous systems starting from integrable autonomous systems. It will be discussed in Sect.4.

Inozemtsev considered degenerations of $U(u|\tau_0)$ playing with the coupling constants, the periods, and u. In this way he obtained the trigonometric, rational and exponential interactions.

Presumably, they describe the degenerations of PVI to PV-PI in terms of degenerations of elliptic functions. Here is one of his potentials:

$$\alpha_0 \frac{1}{\sinh^2 u} + \alpha_1 \frac{1}{\sinh^2 2u} + \alpha_2 \exp u + \alpha_3 \exp 2u.$$

In what follows we consider only the subfamily PVI$_\nu$ corresponding to $\alpha_j = \nu^2$.

3 Isomonodromic deformations

Here we describe the monodromy preserving equations as reduced Hamiltonian systems. The original phase space is infinite-dimensional and almost all degrees of freedom are killed by the symplectic reduction. Our approach differs from [22], where PI-PVI equations are treated as a result of symplectic reduction from finite-dimensional space.

1. Hamiltonian approach

Let Σ_g be a Riemann curve of genus g. Consider the space $FBun_{\Sigma,G}$ of flat vector bundle V_G, where $G = \mathrm{GL}(N, \mathbf{C})$ with smooth connection $\mathcal{A}$. The flatness means that its curvature vanishes

$$F_{\mathcal{A}} = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] = 0. \tag{3.1}$$

Let us fix the complex structure on Σ_g. Then for $\mathcal{A} = (A, \bar{A})$ we have locally a consistent system of matrix differential equations

$$(\partial + A)\Psi = 0,$$

$$(\bar{\partial} + \bar{A})\Psi = 0.$$

We modify this system in the following way. First, introduce formally a parameter $\kappa \in \mathbf{R}$ (*the level*) and consider the operator $\kappa\partial$ instead of ∂ in the first equation. Let μ be a Beltrami differential on Σ_g ($\mu \in \Omega^{(-1,1)}(\Sigma_g)$). It means that in local coordinates $\mu = \mu(z, \bar{z})\frac{\partial}{\partial z} \otimes d\bar{z}$. It allows to deform the complex structure on Σ_g such that the new complex coordinates are

$$w = z - \epsilon(z, \bar{z}), \quad \bar{w} = \bar{z}, \quad \mu(z, \bar{z}) = \frac{\bar{\partial}\epsilon(z, \bar{z})}{1 - \partial\epsilon(z, \bar{z})}.$$

The holomorphic operator $\partial_{\bar{w}} = \bar{\partial} + \mu\partial$ is annihilates the one-form dw likewise $\bar{\partial}$ annihilates dz. We do not touch the anti-holomorphic operator $\kappa\partial$. In the new coordinates (3.1) takes the form

$$F_{\mathcal{A}} = (\bar{\partial} + \partial\mu)A - \kappa\partial\bar{A} + [\bar{A}, A] = 0. \tag{3.2}$$

Thus, we come to the system

$$(\kappa\partial + A)\Psi = 0, \tag{3.3}$$

$$(\bar{\partial} + \mu\partial + \bar{A})\Psi = 0. \tag{3.4}$$

Represent the Beltrami differential as $\mu = \sum_{a=1}^{l} t_a \mu_a^0$, where $\mu_1^0, \ldots, \mu_l^0$ is the basis in the tangent space to the moduli space $\mathcal{M}_g$ of complex structures on Σ_g, ($l = \dim \mathcal{M}_g = 3g - 3$, for $g > 1$, $l = 1$, for $g = 1$). In other words, $\mathbf{t} = (t_1, \ldots, t_l)$ are coordinates of the tangent vector to $\mathcal{M}_g$.

To fix a fundamental solution of (3.3),(3.4), impose the following normalization for some reference point $(z_0, \bar{z}_0) \in \Sigma_g$

$$\Psi(z_0, \bar{z}_0) = I.$$

Let γ be a homotopically nontrivial cycle in Σ_g such that $(z_0, \bar{z}_0) \in \gamma$ and $\mathcal{Y}$ is the corresponding monodromy transformation

$$\mathcal{Y}(\gamma) = \Psi(z_0, \bar{z}_0)|_\gamma = P \exp \oint_\gamma \mathcal{A}.$$

The set of matrices $\{\mathcal{Y}(\gamma)\}$ generates a representation of the fundamental group $\pi_1(\Sigma_g, z_0)$ in $\mathrm{GL}(N, \mathbf{C})$. Independence the monodromy $\mathcal{Y}$ on the deformations of the complex structure means that the linear equations

$$\partial_a \mathcal{Y} = 0, \quad (a = 1, \ldots, l) \; (\partial_a = \partial_{t_a}). \tag{3.5}$$

are consistent with (3.3),(3.4).

Proposition 3.1 *Equations (3.5) are consistent with (3.3),(3.4) iff*

$$\partial_a A = 0, \quad (a = 1, \ldots, l), \tag{3.6}$$

$$\partial_a \bar{A} = \frac{1}{\kappa} A \mu_a^0, \quad (a = 1, \ldots, l). \tag{3.7}$$

The proof is straightforward.

Proposition 3.2 *Equations of motion (3.6),(3.7) are Hamiltonian.*

Endow the space $FBun_{\Sigma,G}$ with the symplectic form

$$\omega^{(0)} = \int_{\Sigma_g} < \delta A, \delta \bar{A} >, \quad (<,>= \mathrm{tr}), \tag{3.8}$$

and the set of Hamiltonians

$$H_a = \frac{1}{2} \int_{\Sigma_g} < A, A > \mu_a^{(0)}, \quad (a = 1, \ldots, l). \tag{3.9}$$

Then (3.6),(3.7) are Hamiltonian equations with respect to $\omega^{(0)}$ and H_s.

Consider the bundle $\mathcal{P}$ over the moduli space $\mathcal{M}_g$ with $FBun_{\Sigma,G}$ as the fibers. The triple $(A, \bar{A}, \mathbf{t})$ can be considered as the local coordinates of the total space of the bundle. It is useful to consider $\mathcal{P}$ as the extended phase space [23]. There is a closed two-form on $\mathcal{P}$

$$\omega = \omega^{(0)} - \frac{1}{\kappa} \sum_a \delta H_a \delta t_a. \tag{3.10}$$

Though ω is degenerated on $\mathcal{P}$ it produces the equations of motion (3.6),(3.7), since the form $\omega^{(0)}$ is non-degenerated along the fibers.

The gauge transformations in the deformed complex structure take the form

$$A \to f^{-1} \kappa \partial f + f^{-1} A f, \quad \bar{A} \to f^{-1} (\bar{\partial} + \mu \partial) f + f^{-1} \bar{A} f. \tag{3.11}$$

The form ω is invariant under these transformations, though its constituents $\omega^{(0)}$ and H_s separately are not invariant.

Introduce a new couple of the connection components $\mathcal{A} = (A, \bar{A}')$, where $\bar{A}' = \bar{A} - \frac{1}{\kappa} \mu A$. In terms of $(A, \bar{A}')$ the form ω (3.10) takes the canonical form

$$\omega = \int_{\Sigma_g} < \delta A, \delta \bar{A}' > . \tag{3.12}$$

2.Symplectic reduction

The form ω is degenerated on $FBun_{\Sigma,G}$, because it is invariant under the action of the group $\mathcal{G}$ of gauge transformations (3.11), generating by the flatness condition (3.2). The gauge fixing along with the flatness condition (3.2) is nothing else as the symplectic reduction from the space of smooth connections $Sm_{\Sigma,G}$ in the bundle V_G to the reduced space

$$\widetilde{FBun}_{\Sigma,G} = FBun_{\Sigma,G}/\mathcal{G} = Sm_{\Sigma,G}//\mathcal{G}$$

The double slashes means that we impose the moment constraints (3.2) and fix the gauge. $\widetilde{FBun}_{\Sigma,G}$ is *the moduli space of flat connections* of the bundle V_G. In terms of the symplectic reduction procedure the flatness condition is called *the moment constraint equation*.

Let us fix the gauge in a such way that the $\bar{A}$ component of $\mathcal{A}$ becomes anti-holomorphic

$$\partial \bar{L} = 0, \quad (\bar{L} = f^{-1}(\bar{\partial} + \mu\partial)f + f^{-1}\bar{A}f). \tag{3.13}$$

We can do it because the antiholomorphity of $f^{-1}(\bar{\partial} + \mu\partial)f + f^{-1}\bar{A}f$ amounts to the classical equations of motion for the Wess-Zumino-Witten functional $S_{WZW}(f, \bar{A})$ for the gauge field f in the external field $\bar{A}$. Denote the gauge transformed field A as L

$$L = f^{-1}\kappa\partial f + f^{-1}Af.$$

Then (3.2) takes the form

$$(\bar{\partial} + \partial\mu)L + [\bar{L}, L] = 0. \tag{3.14}$$

Thus, the moduli space of flat connections $\widetilde{FBun}_{\Sigma,G}$ are characterized by the set of solutions of the linear differential equation (3.14) along with the condition (3.13). The moduli space $\widetilde{FBun}_{\Sigma,G}$ is finite-dimensional space

$$\dim \widetilde{FBun}_{\Sigma,G} = 2(N^2 - 1)(g - 1), \quad g > 1.$$

After the gauge fixing we come to the bundle $\tilde{\mathcal{P}}$ over $\mathcal{M}_g$ with $\widetilde{FBun}_{\Sigma,G}$ as the fibers. The system of linear differential equations (3.3),(3.4) and (3.5) after the gauge fixing takes the form

$$(\kappa\partial + L)\Psi = 0, \tag{3.15}$$

$$(\bar{\partial} + \mu\partial + \bar{L})\Psi = 0, \tag{3.16}$$

$$(\kappa\partial_s + M_s)\Psi = 0, \tag{3.17}$$

where we replaced Ψ on $f^{-1}\Psi$ and $M_s = \kappa\partial_s f f^{-1}$.

The gauge transformations do not spoil the consistency of the system. The consistency (3.15) and (3.16) is provided by (3.14) and (3.13). In fact, the consistency (3.17) with (3.15) and (3.16) leads to the Lax form of the equations of isomonodromic deformations

$$\partial_s L - \kappa\partial M + [M, L] = 0, \tag{3.18}$$

$$\kappa\partial_s \bar{L} - \mu_s^0 L = (\bar{\partial} + \mu\partial)M_s - [M_s, \bar{L}]. \tag{3.19}$$

They play the role (3.6),(3.7) correspondingly. The last equation allows to find M_s in terms of dynamical variables $L, \bar{L}$.

The symplectic form ω on the reduced phase space $\tilde{\mathcal{P}}$ is

$$\omega = \int_{\Sigma_g} < \delta L, \delta\bar{L} > -\frac{1}{\kappa}\sum_s \delta H_s \delta t_s. \tag{3.20}$$

$$H_s = \frac{1}{2}\int_{\Sigma_g} < \delta L, \delta L > \mu_s^{(0)} \tag{3.21}$$

Introduce the local coordinates $(\mathbf{v}, \mathbf{u})$ in $\widetilde{FBun}_{\Sigma,G}$:

$$L = L(\mathbf{v}, \mathbf{u}, \mathbf{t}),\ \ \bar{L} = \bar{L}(\mathbf{v}, \mathbf{u}, \mathbf{t}),$$

$$\mathbf{v} = (v_1, \dots, v_{(N^2-1)(g-1)}),\quad \mathbf{u} = (u_1, \dots, u_{(N^2-1)(g-1)}).$$

Assume for simplicity that this parameterization leads to the canonical form on $\widetilde{FBun}_{\Sigma,G}$

$$\omega^{(0)} = \int_{\Sigma_g} < \delta L(\mathbf{v}, \mathbf{u}, \mathbf{t}), \delta\bar{L}(\mathbf{v}, \mathbf{u}, \mathbf{t}) >= (\delta\mathbf{v}, \delta\mathbf{u}), \tag{3.22}$$

where the pairing in the right hand side is induced by the trace. The form on the extended phase space is

$$\omega = (\delta\mathbf{v}, \delta\mathbf{u}) - \frac{1}{\kappa}\sum_s \delta K_s(\mathbf{v}, \mathbf{u}, \mathbf{t})\delta t_s, \tag{3.23}$$

and the variations of the Hamiltonians K_s in the new variables take the form

$$\delta K_s = \int_{\Sigma_g} [< L, \delta L > \mu_s^{(0)} + \kappa(< \delta L, \partial_s\bar{L} > - < \partial_s L, \delta\bar{L} >)]. \tag{3.24}$$

Now, due to (3.14), the hamiltonians depends explicitly on times. Consider the one-form (the integral invariant of Poincaré-Cartan)

$$\theta = \delta^{-1}\omega = (\mathbf{v}, \delta\mathbf{u}) - \frac{1}{\kappa}\sum_s K_s(\mathbf{v}, \mathbf{u}, \mathbf{t})\delta t_s.$$

There exist $3g-3 = \dim \mathcal{M}_g$-dimensional space of vector fields $\mathcal{V}_s$ that annihilates θ

$$\mathcal{V}_s = \kappa\partial_s + \{H_s, \cdot\},\quad (s = 1, \dots, l). \tag{3.25}$$

It can be checked that $\mathcal{V}_s$ satisfy the following conditions

$$\kappa\partial_s H_r - \kappa\partial_r H_s + \{H_s, H_r\}_{\omega^{(0)}} = 0. \tag{3.26}$$

Thereby, they define the flat connection in the bundle $\tilde{\mathcal{P}}$. These conditions are called *the Whitham hierarchy* (WH). The equations for any function $f(\mathbf{v}, \mathbf{u}, \mathbf{t})$ on $\tilde{\mathcal{P}}$ take the form

$$\frac{df(\mathbf{v}, \mathbf{u}, \mathbf{t})}{dt_s} = \kappa\frac{\partial f(\mathbf{v}, \mathbf{u}, \mathbf{t})}{\partial t_s} + \{H_s, f\} \tag{3.27}$$

They are called *the hierarchy of isomonodromic deformations* (HID).

The both hierarchies can be derived from variations of the prepotential $\mathcal{F}$. It is defined as the integral over the classical trajectories in the extended phase space $\tilde{\mathcal{P}}$

$$\mathcal{F}(\mathbf{u}, \mathbf{t}) = \mathcal{F}(\mathbf{u}_0, \mathbf{t}_0) + \int_{\mathbf{u}_0, \mathbf{t}_0}^{\mathbf{u}, \mathbf{t}} \mathcal{L}_s dt_s, \tag{3.28}$$

where $\mathcal{L}_s(\partial_s\mathbf{u}, \mathbf{u}, \mathbf{t}) = (\mathbf{v}, \partial_s\mathbf{u}) - K_s(\mathbf{v}, \mathbf{u}, \mathbf{t}),\ (\partial_s\mathbf{u} = \frac{\delta K_s}{\delta\mathbf{v}})$ is the Lagrangian. $\mathcal{F}$ satisfies the set of the Hamilton-Jacobi equations

$$\kappa\partial_s\mathcal{F} + H_s(\frac{\delta\mathcal{F}}{\delta\mathbf{u}}, \mathbf{u}, \mathbf{t}) = 0. \tag{3.29}$$

The logarithm of $\mathcal{F}$ is called *the tau-function* of HID.

3. Singular curves

The singular curves are important for applications, since they produce nontrivial systems for the low genus curves ($g = 0, 1$). In these cases the explicit calculations of hamiltonians are available.

Consider a curve $\Sigma_{g,n}$ of genus g with n marked points $(x_1, \ldots, x_n)$. The number of times is equal to dimension of the moduli space $\mathcal{M}_{g,n}$. We extend the space of connections $FBun_{\Sigma,G} = \{A, \bar{A}\}$ by adding the coadoint orbits of $G = \mathrm{GL}(N, \mathbf{C})$ in the marked points

$$(\mathcal{O}_1, \ldots, \mathcal{O}_n), \quad \mathcal{O}_b = \{p_b = gp_b^0 g^{-1}\},$$

where p_b^0 fixes the conjugacy class of $\mathcal{O}_b$. We allow the A component of connection to have simple poles at the marked points, while the Beltrami differentials vanish there. Then instead of (3.2) we obtain

$$(\bar{\partial} + \partial\mu)A - \kappa\partial\bar{A} + [\bar{A}, A] = \sum_{b=1}^{n} \delta^2(x_b)p_b \tag{3.30}$$

The Hamiltonian formalism is provided by the modified symplectic form

$$\omega^{(0)} = \int_{\Sigma_{g,n}} < \delta A, \delta\bar{A} > +2\pi i \sum_{b=1}^{n} < \delta(p_b g_b^{-1}), \delta g_b > . \tag{3.31}$$

To derive HID one should start from the space of connections with simple poles in the marked points.

Finally, we come to the same linear system (3.15), (3.16),(3.17), but due to (3.30) the following relation between L and $\bar{L}$ holds

$$(\bar{\partial} + \partial\mu)L + [\bar{L}, L] = \sum_{b=1}^{n} \delta^2(x_b)p_b. \tag{3.32}$$

As before, the linear equations are equivalent to the equations of motion of HID coming from the symplectic form

$$\omega = \omega^{(0)}(\mathbf{v}, \mathbf{u}, \mathbf{p}) - \frac{1}{\kappa}\sum_{s=1}^{l} \delta K_s(\mathbf{v}, \mathbf{u}, \mathbf{p}, \mathbf{t})\delta t_s, \quad (l = \dim(\mathcal{M}_{g,n}), \tag{3.33}$$

where $\mathbf{p} = (p_1, \ldots, p_n)$,$\omega_0$ is determined by the reduction from (3.31)

$$\omega^{(0)} = \int_{\Sigma_{g,n}} < \delta L(\mathbf{v}, \mathbf{u}, \mathbf{p}), \delta\bar{L}(\mathbf{v}, \mathbf{u}, \mathbf{p}) > +2\pi i \sum_{b=1}^{n} < \delta(p_b g_b^{-1}), \delta g_b >, \tag{3.34}$$

and K_s (3.24).

4 Hitchin systems and their Whitham deformations

1. Hitchin systems

Consider the moduli space $\mathcal{R}_{g,N}$ of stable holomorphic $\mathrm{GL}(N, \mathbf{C})$ vector bundles V over Σ_g. It is a smooth variety of dimension

$$\dim \mathcal{R}_{g,N} = \tilde{g} = N^2(g-1) + 1. \tag{4.1}$$

Let $T^*\mathcal{R}_{g,N}$ be the cotangent bundle to $\mathcal{R}_{g,N}$ with the standard symplectic form on it. Hitchin [10] defined a completely integrable system on $T^*\mathcal{R}_{g,N}$.

The space $T^*\mathcal{R}_{g,N}$ can be obtained by the symplectic reduction from the space $T^*\mathcal{R}^s_{g,N} = (\Phi, \bar{A})$, where $\bar{A}$ is a smooth connection of the stable bundle corresponding to $\bar{\partial} + \bar{A}$ and Φ is the *Higgs field* $\Phi \in \Omega^0(\Sigma_g, EndV \otimes K)$ (K is the canonical bundle of Σ_g). There is the well defined symplectic form on this space

$$\omega^{(0)} = \int_{\Sigma_g} < \delta\Phi, \delta\bar{A} > . \tag{4.2}$$

This form is invariant with respect to the gauge group $\mathcal{G} = C^\infty Map(\Sigma_g, \mathrm{GL}(N, \mathbf{C}))$ action

$$\Phi \to f^{-1}\Phi f, \quad \bar{A} \to f^{-1}\bar{\partial} f + f^{-1}\bar{A} f. \tag{4.3}$$

In particular, $\mathcal{R}_{g,N} = \mathcal{R}^s_{g,N}/\mathcal{G}$. Let $\rho_{s,k} = \rho_{s,k}\partial_z^{k-1} \otimes d\bar{z}$ be the $(-k+1,1)$-differentials ($\rho_{s,k} \in H^1(\Sigma_g, \Gamma^{k-1} \otimes K)$), and s enumerates the basis in $H^1(\Sigma_g, \Gamma^{k-1} \otimes K)$. ($\rho_{s,2} = \mu_s$). Due to the Riemann-Roch theorem

$$\dim H^1(\Sigma_g, \Gamma^{k-1} \otimes K) = (2k-1)(g-1).$$

These differentials allows to define the gauge invariant Hamiltonians

$$H_{s,k} = \frac{1}{k}\int_{\Sigma_g} < \Phi^k > \rho_{s,k}, \quad (k = 1, \dots, N, \; s = 1, \dots, (2k-1)(g-1)). \tag{4.4}$$

The Hamiltonian equations take the form

$$\partial_a \Phi = 0, \quad (\partial_a = \frac{\partial}{\partial t_a}, a = (s,k)), \tag{4.5}$$

$$\partial_a \bar{A} = \Phi^{k-1}\rho_{s,k} \tag{4.6}$$

The gauge action produces the moment map $\mu : \; T^*\mathcal{R}^s \to Lie^*(\mathrm{GL}(N, \mathbf{C}))$. It follows from (4.2),(4.3) that $\mu = \bar{\partial}\Phi + [\bar{A}, \Phi]$. The reduced phase space is the cotangent bundle we started with

$$T^*\mathcal{R}_{g,N} \sim T^*\mathcal{R}^s//\mathcal{G} := \mu^{-1}(0)/\mathcal{G}.$$

The *Hitchin hierarchy* (HH) is the set of Hamiltonian equations with $H_{s,k}$ (4.4) on the reduced phase space $T^*\mathcal{R}_{g,N}$. Hitchin observed that the number of integrals $H_{s,k}$

$$\sum_{k=1}^{N}(2k-1)(g-1) = N^2(g-1) + 1$$

coincides with dimension $\tilde{g}$ of the coordinate space $\mathcal{R}_{g,N}$ (4.1). Since they are independent and Poisson-commute, HH is the set of completely integrable Hamiltonian systems.

Let fix the gauge of the field $\bar{A}$

$$\bar{A} = f\bar{\partial} f^{-1} + f\bar{L} f^{-1}.$$

Then

$$L = f^{-1}\Phi f.$$

is a solution of the moment constraint equation

$$\bar{\partial} L + [\bar{L}, L] = 0. \tag{4.7}$$

The space of solutions of this equation is isomorphic to $H^0(\Sigma_g, End\ V \otimes K)$ - the cotangent space to the moduli space $\mathcal{R}_{g,N}$.

The gauge transformation f defines the element M_a in $\mathrm{Lie}(\mathrm{GL}(N, \mathbf{C}))$ $M_a = \partial_a f f^{-1}$.

Proposition 4.1 *The system of linear equations*

$$(\lambda + L)Y = 0, \tag{4.8}$$

$$(\bar{\partial} + \sum_{s,k} \lambda^{k-1} t_{s,k}\rho_{s,k} + \bar{L})Y = 0, \tag{4.9}$$

$$(\partial_a + M_a)Y = 0 \tag{4.10}$$

is consistent and defines the equations of motion for HH.

Proof. The consistency of (4.8) and (4.9) follows from (4.7). In terms of L the equations of motion (4.5),(4.6) take the form

$$\partial_a L + [M_a, L] = 0 \text{ (the Lax equation)}, \tag{4.11}$$

$$\partial_a \bar{L} - \bar{\partial} M_a + [M_a, \bar{L}] = L^{k-1}\rho_{s,k}, \quad (a = (s,k)). \tag{4.12}$$

The Lax equation provides the consistency of (4.8) and (4.10), while the second equation of motion (4.12) plays the same role for the couple (4.9) and (4.10). This equation allows to determine M_a from L and $\bar{L}$.

The bundles over singular curves can be incorporated in this approach as well [24]. The Higgs field has simple poles at the marked points and (4.7) is replaced by

$$\bar{\partial} L + [\bar{L}, L] = 2\pi i \sum_{b=1}^{n} \delta^2(x_b) p_b. \tag{4.13}$$

The form $\omega^{(0)}$ on $T^*\mathcal{R}_{g,N}$

$$\omega^{(0)} = \int_{\Sigma_g} < \delta L, \delta \bar{L} > + 2\pi i \sum_{b=1}^{n} < \delta(p_b g_b^{-1}), \delta g_b > \tag{4.14}$$

Note that the space $T^*\mathcal{R}_{g,N}$ and the space of flat bundles $\widetilde{FBun}_{\Sigma,G}$ have the same dimensions. Moreover, it follows from the comparison the moment constraints (3.32),(4.13) and the symplectic forms (3.34),(4.14) that they are isomorphic as symplectic manifolds.

2.Spectral description

Due to the Liouville theorem the phase flows of HH are restricted to the Abelian varieties, corresponding to a level set of the Hamiltonians $H_{s,k} = c_{s,k}$. The phase flow takes a simple form in terms of action-angle coordinates. They are defined in a such way that the angle type coordinates are angular coordinates on the Abelian variety, and the hamiltonians depend on the action coordinates only. To describe them consider the characteristic polynomial of the matrix L

$$P(\lambda, z) = \det(\lambda + L) = \lambda^N + b_1\lambda^{N-1} + \ldots + b_j\lambda^{N-j} + \ldots + b_N, \tag{4.15}$$

$$b_j = \sum Min_j, \quad (Min_j - \text{principle minors of order } j, \ b_1 = \mathrm{tr}L, \ b_N = \det L).$$

The spectral curve $\mathcal{C} \subset T^*\Sigma_g$ is defined as the zero set of P

$$C = \{P(\lambda, z) = 0\}.$$

It is a well defined object, because the coefficients b_j are gauge invariant.

Since $L \in H^0(\Sigma_g, End\ V \otimes K)$, the coefficients $b_j \in H^0(\Sigma_g, K^j)$ and we obtain the map

$$p : T^*\mathcal{R}_{g,N} \to B = \oplus_{j=1}^N H^0(\Sigma_g, K^j). \tag{4.16}$$

The space B can be considered as the moduli space of the family of spectral curves parameterized by the Hamiltonians $H_{s,k}$. The fibers of p are Lagrangian subvarieties of $T^*\mathcal{R}_{g,N}$. The spectral curve $\mathcal{C}$ is the N-fold covering of the basis curve $\Sigma_{g,N}$

$$\pi : C \to \Sigma_{g,N}.$$

Its genus $g(\mathcal{C})$ is equal to dimension $\tilde{g}$ of $\mathcal{R}_{g,N}$. There is a line bundle $\mathcal{L}$ with an eigenspace of $L(z)$ corresponding to the eigenvalue λ as a fiber over a generic point (λ, z)

$$\mathcal{L} \subset \ker(\lambda + L) \subset \pi^*(V).$$

It defines a point of the Jacobian $Jac(\mathcal{C})$, the Liouvillean variety of dimension $\tilde{g} = g(\mathcal{C})$.

Conversely, if $z \in \Sigma_g$ is not a branch point one can reconstruct V for a given line bundle on $\mathcal{C}$ as

$$V_z = \oplus_{v \in \pi^{-1}(z)} \mathcal{L}_v.$$

Let $\omega_j,\ j = 1 \ldots, \tilde{g}$ be the canonical holomorphic one-differentials on $\mathcal{C}$ such that for the cycles $\alpha_1, \ldots, \alpha_{\tilde{g}}; \beta_1, \ldots, \beta_{\tilde{g}},\quad \alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0,\quad \alpha_i \cdot \beta_j = \delta_{ij},\quad \oint_{\alpha_i} \omega_j = \delta_{ij}$. Then the symplectic form $\omega^{(0)}$ (4.2) can be written in the form

$$\omega^{(0)} = \int_{\Sigma_g} < \delta L, \delta \bar{L} > = \sum_{j=1}^N \int_{\Sigma_g} \delta\lambda_j \delta\xi_j.$$

Here ξ_j are diagonal elements of $s\bar{L}s^{-1}$, where s diagonalizes L, $sLs^{-1} = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$. Then we obtain

$$\omega^{(0)} = \int_{\mathcal{C}} \delta\lambda\delta\xi.$$

Because λ is a holomorphic one-form on $\mathcal{C}$, it can be decomposed as $\lambda = \sum_{j=1}^{\tilde{g}} a_j \omega_j$. Thereby

$$\omega^{(0)} = \sum_{j=1}^{\tilde{g}} \delta a_j \int_{\mathcal{C}} \omega_j \delta\xi.$$

The action variables can be identify with

$$a_j = \oint_{\alpha_i} \lambda\omega_j, \quad (j = 1 \ldots, \tilde{g}). \tag{4.17}$$

To define the angle variables, put locally $\xi = \bar{\partial} \log \psi$. If (p_m) is a divisor of ψ then

$$\int_{\mathcal{C}} \omega_j \delta\xi = \sum_m \int_{p_0}^{p_m} \omega_j \log \psi = \delta\varphi_j.$$

Thus φ_j are linear coordinates on $Jac(\mathcal{C})$ and

$$\omega^{(0)} = \sum_{j=1}^{\tilde{g}} \delta a_j \delta\varphi_j.$$

3. Scaling limit.

Consider HID in the limit $\kappa \to 0$. The value $\kappa = 0$ is called critical. We prove that on the critical level HID coincide with part of HH relating to the quadratic Hamiltonians (4.4).

Note first, that in this limit the A-connection is transformed in the Higgs field Φ: $A \stackrel{\kappa\to 0}{\longrightarrow} \Phi$, and therefore

$$\widetilde{FBun}_{\Sigma,G} \stackrel{\kappa\to 0}{\longrightarrow} T^*\mathcal{R}_{g,N}.$$

But the form ω on the extended phase $\mathcal{P}$ appears to be singular (see (3.10),(3.20)). To get around we rescale the times

$$\mathbf{t} = \mathbf{T} + \kappa\mathbf{t}^{\mathbf{H}}, \tag{4.18}$$

where $\mathbf{t}^H$ are the fast (Hitchin) times and $\mathbf{T}$ are the slow times. Assume that only fast times are dynamical. It means that

$$\delta\mu(\mathbf{t}) = \kappa\sum_s \mu_s^{(0)}\delta t_s^H, \quad (\mu_s^{(0)} = \bar{\partial}n_s).$$

After this rescaling the forms (3.10),(3.20) become regular. The rescaling procedure means that we blow up a vicinity of the fixed point $\mu_s^{(0)}$ in $\mathcal{M}_{g,n}$ and the whole dynamic is developed in this vicinity. This fixed point is defined by the complex coordinates

$$w_0 = z - \sum_s T_s\epsilon_s(z,\bar{z}), \quad \bar{w}_0 = \bar{z}. \tag{4.19}$$

Now compare the Baker-Akhiezer function of HID Ψ (3.15), (3.16),(3.17) with the Baker-Akhiezer function of HH Y (4.8),(4.9),(4.10). Using the WKB approximation, assume that

$$\Psi = \Phi\exp(\frac{\mathcal{S}^{(0)}}{\kappa} + \mathcal{S}^{(1)}), \tag{4.20}$$

where Φ is a group valued function and $\mathcal{S}^{(0)}$, $\mathcal{S}^{(1)}$ are diagonal matrices. Let substitute (4.20) in the linear system (3.15),(3.16),(3.17). If

$$\frac{\partial}{\partial\bar{w}_0}\mathcal{S}^{(0)} = 0, \quad \frac{\partial}{\partial t_s^H}\mathcal{S}^{(0)} = 0.$$

there are no terms of order κ^{-1}. It follows from the definition of the fixed point in the moduli of complex structures (4.19) that

$$\mathcal{S}^{(0)} = \mathcal{S}^{(0)}(T_1,\ldots,T_l|z - \sum_s T_s\epsilon_s(z,\bar{z})). \tag{4.21}$$

We take also $\mathcal{S}^{(1)} = \partial\mathcal{S}^{(0)}\sum_s t_s^H\epsilon_s(z,\bar{z})$. In the quasi-classical limit we put

$$\partial\mathcal{S}^{(0)} = \lambda. \tag{4.22}$$

In the zero order approximation we come to the linear system of HH (3.15),(3.16),(3.17), defining by the Hamiltonians $H_{k,s},\ k = 1,2$. The Baker-Akhiezer function Y takes the form

$$Y = \Phi e^{\sum_s t_s^H\frac{\partial}{\partial T_s}\mathcal{S}^{(0)}}. \tag{4.23}$$

Our goal is the inverse problem. We need to reconstruct the dependence on the slow times $\mathbf{T}$ starting from solutions of HH. Since $\mathbf{T}$ is a vector in the tangent space to the moduli of curves

$\mathcal{M}_{g,n}$, it defines a deformation of the spectral curve in the space B (see (4.4),(4.16)). Solutions Y of the linear systems (4.8)(4.9),(4.10) take the form $Y = \Phi e^{\sum_s t_s^H \Omega_s}$, where Ω_s are diagonal matrices. Their entries are primitive functions of meromorphic differentials with singularities matching the corresponding poles of L. Then according with (4.23) we can assume that

$$\frac{\partial}{\partial T_s} d\mathcal{S} = d\Omega_s.$$

These equation define the approximation to the phase of Ψ in the linear problems (3.15),(3.16), (3.17) of HID along with

$$\frac{\partial}{\partial a_j} d\mathcal{S} = \omega_j.$$

The differential $d\mathcal{S}$ plays role of the Seiberg-Witten differential. Important point is that only part of the spectral moduli, connected with $H_{k,s}$, $k = 1, 2$, is deformed. As a result there is no matching between the action parameters of the spectral curve $a_j, j = 1, \dots, \tilde{g}$ (4.17) and deformed hamiltonians. The detailed analyses of this situation in the rational case is undertaken in [25].

Another object of the Whitham quantization is the prepotential $\mathcal{F}$ (3.28). It depends on the action variables a_j . This dependence is compatible with the Hamilton-Jacobi equation (3.29) with slow times T_s as the independent variables. These equations are discussed in [25, 26].

5 Classical limit of the Knizhnik-Zamolodchikov-Bernard equations

The Knizhnik-Zamolodchikov-Bernard equations (KZB) are the system of differential equations having the form of the non-stationer Schrödinger equations with the times coming from $\mathcal{M}_{g,n}$ (see, for example, [27]).

They arise in the geometric quantization of the moduli of flat bundles $\widetilde{FBun}_{\Sigma,G}$ [28, 29]. Let $V = V_1 \times \cdots \otimes V_n$ be the tensor product of finite-dimensional irreducible representations associated with the marked points. The Hilbert space of the quantum system is a space of sections of the bundle $\mathcal{E}_{V,\kappa^{quant}}(\Sigma_{g,n})$ over $\widetilde{FBun}_{\Sigma,G}$ depending on an negative number κ^{quant} with the V-fibers. It is the space of conformal blocks of the WZW theory on $\Sigma_{g,n}$.

The Hitchin systems are the classical limit of the KZB equations on the critical level [24, 27]. The classical limit means that one replaces operators by their symbols and generators of finite-dimensional representations in the vertex operator acting in the spaces V_j by the corresponding elements of coadjoint orbits. To pass to the classical limit in the KZB equations

$$(\kappa^{quant}\partial_s + \hat{H}_s)F = 0. \tag{5.1}$$

we replace the conformal block by its quasi-classical expression

$$F = \exp \frac{\mathcal{F}}{\hbar}, \tag{5.2}$$

where $\hbar = (\kappa^{quant})^{-1}$. Consider the classical limit $\kappa^{quant} \to \infty$ and assume that values of the Casimirs C_a^i, $(i = 1, \dots, \text{rank}G,\ a = 1, \dots, n)$ corresponding to the irreducible representations defining the vertex operators also go to infinity. Let all values $\lim \frac{C_a^i}{\kappa^{quant}}$ are finite. It allows to fix the coadjoint orbits in the marked points. In the classical limit (5.1) is transformed to the Hamilton-Jacobi equation for the action $\mathcal{F} = \log \tau$ (3.29) of HID.

The integral representations of conformal blocks are known for WZW theories over rational and elliptic curves [31, 32, 33, 34]. Then (5.2) allows to extract the prepotential $\mathcal{F}$ of HID.

The KZB operators (5.1) play role of flat connections in the bundle $\mathcal{P}^{quant}$ over the moduli of curves $\mathcal{M}_{g,n}$ with the fibers $\mathcal{E}_{V,\kappa^{quant}}(\Sigma_{g,n})$ [4, 30]

$$[\kappa^{quant}\partial_s + \hat{H}_s, \kappa^{quant}\partial_r + \hat{H}_r] = 0.$$

These equations is the quantum counterpart of the Whitham hierarchy (3.26).

6 Multicomponent generalization of PVI$_\nu$

Consider $\widetilde{FBun}_{\Sigma,G}$ over the family of elliptic curves with a one marked point $\mathcal{M}_{1,1}$. The space $\mathcal{M}_{1,1}$ is one-dimensional, because the position of one point on a torus is irrelevant. Thus, we have only one time τ and $\mathcal{M}_{1,1} \sim E_\tau$ (2.1). In this case the Beltrami differential takes the form

$$\mu = \frac{\tau - \tau_0}{\tau - \bar{\tau}_0}.$$

Consider the most degenerated orbit $\mathcal{O} = (gp^0g^{-1})$ of GL$(N, \mathbf{C})$ sitting in the marked point $z = 0$ with

$$p^0 = \nu[(\underbrace{1, \dots, 1}_{N})^T \otimes (\underbrace{1, \dots, 1}_{N}) - Id]. \tag{6.1}$$

For stable bundles the gauge transforms allow to put $\bar{A}$ component in the diagonal form

$$\bar{L} = \frac{2\pi i}{\tau - \bar{\tau}_0}\mathbf{u}, \quad \mathbf{u} = \mathrm{diag}(u_1, \dots, u_N) \in \mathcal{H} - \text{Cartan algebra}. \tag{6.2}$$

It means that

$$\int_{E_\tau} \bar{L} dw d\bar{w} = \mathbf{u}. \tag{6.3}$$

Let $\bar{L} = \bar{\partial}\log\phi$. Then the integral

$$\int_{E_\tau} \bar{L} dw d\bar{w} = \int_{P_0}^{P} \log\phi dw.$$

defines the Abel map E_τ in the product of N Jacobians.

The remaining gauge transforms do not change the gauge fixing. These transformations are generated by the Weyl subgroup W of G and elements $f(w, \bar{w}) \in \mathrm{Map}(T^2_\tau, \mathrm{Cartan}(G))$. The orbit variables can be gauged away by these transforms and we are left with $p^{(0)}$ (6.1). The solution L of the moment constraint

$$\partial_{\bar{w}} L + [\bar{L}, L] = 2\pi i \delta^2(0) p^{(0)}$$

takes the form

$$L = P + X, \;\; P = 2\pi i(\frac{\mathbf{v}}{1-\mu} - \kappa\frac{\mathbf{u}}{\rho}), \tag{6.4}$$

$$\mathbf{u} = \mathrm{diag}(u_1, \dots, u_N), \quad \mathbf{v} = \mathrm{diag}(v_1, \dots, v_N)$$

$$X_{jk} = x(u_j - u_k) = (\tau - \bar{\tau}_0)\nu \exp 2\pi i\{\frac{w - \bar{w}}{\tau - \bar{\tau}_0}(u_j - u_k)\}\phi(\alpha(u_j - u_k), w),$$

$$\phi(u,z) = \frac{\theta(u+z)\theta'(0)}{\theta(u)\theta(z)}, \quad \theta(z|\tau) = q^{\frac{1}{8}} \sum_{n\in\mathbf{Z}} (-1)^n e^{\pi i(n(n+1)\tau+2nz)}.$$

The operator M defining the phase flow according with the Lax equation (3.18) can be extracted from (3.19)

$$M = -D + Y, \ D = \mathrm{diag}(d_1,\dots,d_N), \quad d_j = \sum_{i\neq j}^{N} s(u_j - u_i), \ s(u) = \frac{1}{\kappa}\wp(u) + const. \tag{6.5}$$

$$Y_{jk} = y(u_j - u_k), \quad y(u,w,\bar{w}) = \frac{\rho}{2\pi i\kappa(\tau - \bar{\tau}_0)}\partial_u x(u,w,\bar{w}).$$

The functions x, y, z satisfy the functional equations

$$x(u,z,\bar{z})y(v,z,\bar{z}) - x(v,z,\bar{z})y(u,z,\bar{z}) = (s(v) - s(u))x(u+v,z,\bar{z}). \tag{6.6}$$

This equation is derived from the Lax equation (3.18).

The symplectic form ω is boiled down to

$$\omega = (\delta\mathbf{v}, \delta\mathbf{u}) - \frac{1}{\kappa}\delta H\delta\tau,$$

where

$$H = \frac{(\delta\mathbf{v}, \delta\mathbf{v})}{2} - \frac{\nu^2}{(2\pi i)^2}\sum_{j<k}^{N}\wp(u_j - u_k|\tau).$$

They define the hamiltonian flow

$$\frac{d^2 u_j}{d\tau^2} = \frac{\nu^2}{(2\pi i)^2}\sum_{k<j}^{N}\wp_u(u_j - u_k|\tau). \tag{6.7}$$

For$N = 2$ one can put $u_1 = -u_2 = u$. Then the potential

$$\frac{\nu^2}{(2\pi i)^2}\wp(2u|\tau) = \frac{\nu^2}{(2\pi i)^2}\sum_{j=0}^{3}\wp(u + \frac{T_j}{2}|\tau)$$

produces PVI_ν (see (2.7).

The remaining gauge symmetries implies that ω is invariant under the Weyl transformations W of $(\mathbf{v}, \mathbf{u})$ and the lattice actions (compare with (2.4)

$$\mathbf{v} \to s\mathbf{v}, \ \mathbf{v} + \kappa\mathbf{n}, \ \mathbf{u} \to s\mathbf{u}, \ \mathbf{u} - \mathbf{m} + \tau\mathbf{n}, \ (s \in W, \ \mathbf{n} \in \mathbf{Z}^N).$$

It is also invariant under the $\mathrm{PSL}_2(\mathbf{Z})$ action on τ

$$\tau \to \frac{a\tau + b}{c\tau + d}, \ \mathbf{v} \to \mathbf{v}(c\tau + d) - \kappa c\mathbf{u}, \ \mathbf{u} \to \mathbf{u}(c\tau + d)^{-1}.$$

This invariance is follows from the invariance of the upstairs system under the diffeomorphisms of $\Sigma_{g,n}$ (see [14] for detailes).

On the critical level $\kappa \to 0$, $\tau - \tau_0 = \kappa t$ we obtain the elliptic Calogero N-body system. This system is a particular example of the Hitchin systems [24]. Note, that the functions x, y and s defining the Lax matricies satisfy the same functional equation (6.6) as in the Calogero-Hitchin limit $\kappa = 0$ [18].

7 Conclusion

Here we propose a few open problems in the context of topics discussed above.

• The evident problem is a description of PVI with four arbitrary constants as a reduced Hamiltonian system. The first step in this direction is the Lax form of $PVI_{\alpha,\beta,\gamma,\delta}$. The Lax form is even unknown on the critical level, i.e. for the Calogero-Inozemtsev system. It will be interesting to generalize this approach to the N-body Calogero-Inozemtsev system and the N-component PVI with four coupling constants.

• The degenerations of PVI to PV-PI in terms of elliptic functions.

• There exists a generalization of the Calogero systems related to any simple group. In addition to degrees of freedom coming from the moduli of bundles (the coordinates of particles), these systems certainly contain degrees of freedom related to the coadjoint orbits. Recently a new Lax equations based on arbitrary root systems without the orbit coordinates were proposed [36, 37]. This construction is purely algebraic and does not use the symplectic reduction. How these systems can be incorporated in the Hitchin approach, or, more generally, in the isomonodromic deformation construction?

• Consider the $N = 2$ elliptic Calogero system. The solution $u(t)$, corresponding to the fixed value h_2 of the Hamiltonian

$$H = \frac{v^2}{2} + \frac{\nu^2}{4\pi^2}\wp(2u|\tau_0) = h_2,$$

is implicitly described by the elliptic integral of the first kind

$$t - t_0 = \frac{1}{2}\int_{2u_0}^{2u} \frac{dx}{y'}, \quad y' = y\sqrt{2h_2 - \frac{\nu^2}{2\pi^2}},$$

where $y = 4(x - e_1(\tau_0))(x - e_2(\tau_0))(x - e_3(\tau_0))$. As it was mentioned at the end of Sect.6 it can serve for the calculations of solutions to PVI_ν. This procedure can be accomplished by the Krichever averaging method [35]. It will be interesting to compare this approximation with explicit solutions presented recently in [38, 39] for some particular value of the coupling constant ν.

As suggested in Sect.5 another way of approximation comes from the classical limit of conformal blocks for $SL_2(\mathbf{C})$ theory on elliptic curves with one marked point [33, 40]. Which method gives the better approximation?

• We considered deformations with respect to the moduli of complex structures of curves. They describe only part of the moduli of the spectral curves $\mathcal{C}$. The remaining moduli of $\mathcal{C}$ come from $\rho_{s,k},\ k > 2$. They correspond to the so-called W-geometry of the basic curve $\Sigma_{g,n}$. This geometry is poorly understood. On the other side, there are no examples of isomonodromic deformation equations with respect to these moduli spaces, as well as the corresponding higher order KZB equations. Any progress in understanding of one of these subjects will shed light on another.

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References

[1] B. Gambier, *Sur les équations differentielles du second ordre et du premier degré dont l'integral générale a ses points critiques fixes*, Acta Math. Ann **33** (1910) 1-55

[2] *Workshop on Painlevé Transcedents, Their asymptotics and Physical Applications*, NATO ASI Ser. B: Physics Vol. 278 (1990) Sainte Adele, Quebec, ed. by D. Levi and P. Winternitz

[3] B.Dubrovin, *Integrable systems in topological field theories*, Nucl. Phys. **B 379** (1992) 627-689

[4] N.Hitchin, *Twistors spaces, Einstein metrics and isomonodromic deformations*, Journ. Diff. Geom. **3** (1995) 52-134

[5] D.Korotkin and V.Matveev, *On solutions of the Schlesinger system and Ernst equation in terms of theta-functions*, Preprint AEI-087, 1998

[6] E.D'Hoker and D. H. Phong,*Calogero-Moser Systems in SU(N) Seiberg-Witten Theory*, Nucl.Phys. **B513** (1998) 405-444

[7] R. Fuchs, *Über lineare homogene Differentialgleichungen zweiter ordnung mit im endlich gelegne wesentlich singulären Stellen*, Math. Annalen, **63** (1907) 301-323

[8] J.Malmquist, *Sur les éuations différentielles du second odre dont l'intégrale générale a ses points critiques fixes*, Ark. Mat. Astr. Fys. **17** (1922/23), 1-89

[9] K.Iwasaki *Fuchsian moduli on a Riemann surface - its Poisson structure and Poincaré-Lefschetz duality*, Pacific J. Math. **155** (1992) 319-340

[10] N. Hitchin, *Stable bundles and Integrable Systems*, Duke Math. Journ., **54** (1987) 91-114

[11] I. Krichever, *The tau-function of the universal Whitham hierarchy, matrix models and topological field theories*, Comm. on Pure and Appl. Math. **XLVII** (1994) 437-475

[12] V. Knizhnik and A. Zamolodchikov, *Current algebra and Wess-Zumino model in two dimensions*, Nucl.Phys. **B247** (1984) 83

[13] D. Bernard, *On the Wess-Zumino-Witten models on the torus*, Nucl.Phys. **B303** (1988) 77; *On the Wess-Zumino-Witten models on the Riemann surfaces*, Nucl.Phys. **309** (1988) 145

[14] A.Levin and M.Olshanetsky, *Isomonodromic deformations and Hitchin systems*, Amer. Math. Soc. Transl. (2) (1999)

[15] Painlevé, *Sur les équations différentielles du second odre à points critics fixes*, CRAS, **143** (1906) 1111-1117

[16] Yu.I.Manin, *Sixth Painlevé equation, universal elliptic curve, and mirror of P^2*, Preprint Max-Planck-Institut (1996), alg-geom/9605010, Amer. Math. Soc. Transl. (2) **186** (1998) 131-151

[17] A.Babich and L.Bordag, *The elliptic form of the sixth Painlevé equation,* Preprint NTZ 25/1997

[18] F.Calogero, *Exactly solvable one-dimensional many-body problems*, Lett. Nuovo Cim. **13** (1976) 411-417

[19] V.Inozemtsev, *Lax Representation with spectral parameter on a torus for integrable particle systems*, Lett. Math. Phys. **17** (1989) 11-17

[20] M.G.Darboux, *Sur une équation linéare* Comptes Rendues de l'Academie de Science de Paris, **XCIV** N25 (1882) 1645-1648

[21] A.Treibich, J.-L.Verdier, *Revêtements tangentiels et sommes de 4 nombres triangulaires*, C.R. Ac. Sci. Paris, sér. Math., **311** (1990), 51-54

[22] J. Harnad, *Dual Isomonodromic Deformations and Moment Maps to Loop Algebras*, Commun. Math. Phys. **166** (1994) 337-366; J. Harnad and M.-A. Wisse, *Loop Algebra Moment Maps and Hamiltonian Models for the Painleve Transcendants,* Fields Inst. Commun. **7** 155-169 (1996)

[23] V.I.Arnold, *Mathematical methods of classical mechanics*, Springer-Verlag, NY

[24] N. Nekrasov, *Holomorphic bundles and many-body systems*, PUPT-1534, Comm. Math. Phys., **180** (1996) 587-604; hep-th/9503157

[25] K.Takasaki, *Spectral Curves and Whitham Equations in the Isomonodromic Problems of Schlesinger Type* solv-int/9704004

[26] H.Itoyama and A.Morozov, *Prepotential and the Seiberg-Witten Theory*, Nucl.Phys. **B491** (1997) 529-573

[27] D.Ivanov, *KZB eqs. as a quantization of nonstationary Hitchin systems*, hep-th/9610207

[28] S. Axelrod, S. Della Pietra, and E. Witten, *Geometric quantization of the Chern-Simons gauge theory*, Journ. Diff. Geom., **33** (1991) 787-902

[29] N. Hitchin, *Flat connections and geometric quantization*, Comm.Math.Phys., **131** (1990) 347-380

[30] G.Felder, *The KZB equations on Riemann surfaces*, hep-th/9609153, to appear in the Proceedings of the 1995 les Houches Summer School

[31] V.Schechtman and A.Varchenko, *Arrangements of hyperplanes and Lie algebra homology*, Inv.Math. **106** (1991) 139-194

[32] F.Falceto and K.Gawedzky, *Elliptic Wess-Zumino-Witten model from elliptic Chern-simons theory,* hep-th/9502161, Lett.Math.Phys. **38** (1996) 155

[33] P.Etingof and A.Kirillov, *Representations of affine Lie algebras, parabolic differential equations and Láme functions*, Duke Math. J. **74** (1994), 585-614

[34] G.Felder and C.Weiczerkovski, *Conformal blocks on elliptic curves and Knizhnik-Zamolodchikov-Bernard equations*, hep-th/9411004, Comm.Math.Phys. **176** (1996) 113

[35] I.Krichever, *Method of averaging for two-dimensional "integrable" equations*, Func. Anal. Appl. **22** (1988), 200-213

[36] D.H.Phong, Talk given at this conference; E. D'Hoker and D. H. Phong, *Calogero-Moser Lax Pairs with Spectral Parameter for General Lie Algebras*, Nucl.Phys. **B530** (1998) 537-610

[37] R.Sasaki, Talk given at this conference; A. J. Bordner, E. Corrigan, and R. Sasaki, *Calogero-Moser Models: A New Formulation* hep-th/9805106

[38] P.Deift, A.Its, A.Kapaev, and X.Zhou, *On the algebro-geometric integration of the Schlesinger equations and on elliptic solutions of the Painlevé VI equation*, Preprint IUPUI 98-2 (1998).

[39] A.Kitaev and D. Korotkin, *On solutions of the Schlesinger Equations in Terms of* Θ*-Functions*, math-ph/9810007

[40] G.Felder and A.Varchenko, *Integral representation of solutions of the elliptic Knizhnik–Zamolodchikov–Bernard equations*, hep-th/9502165, Int. Math. Res. Notices (1995) 221-233